Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **14** (2024) 27–30
DOI: 10.47443/dml.2024.044

Research Article

Tropical geometry of Rado matroids

Calum Buchanan*, Richard Danner

Department of Mathematics & Statistics, University of Vermont, Burlington, VT, USA

(Received: 19 February 2024. Received in revised form: 8 July 2024. Accepted: 26 July 2024. Published online: 1 August 2024.)

Abstract

In this article, we characterize the products of simplicial generators for the Chow ring of a loopless matroid, extending a result of Backman, Eur, and Simpson [*J. Eur. Math. Soc.*, DOI: 10.4171/JEMS/1350]. We prove that the stable intersection of a collection of tropical hyperplanes centered at the origin with the Bergman fan of a matroid is the Bergman fan of the dual of a certain Rado matroid.

Keywords: tropical geometry; Chow ring; Rado matroid.

2020 Mathematics Subject Classification: 05B35, 05B40, 14T05, 14C17.

1. Introduction

The Chow ring $A^\bullet(M)$ of a loopless matroid M on ground set E was introduced by Feichtner and Yuzvinsky in [5] as a generalization of the cohomology ring of De Concini and Procesi's wonderful compactification of the complement of a hyperplane arrangement [3]. The importance of the Chow ring was demonstrated by Adiprasito, Huh, and Katz in the proof of the Heron-Rota-Welsh conjecture [1]. Feichtner and Yuzvinsky define $A^\bullet(M)$ to be the graded ring $\mathbb{R}[z_F \mid F \in \mathscr{L}_M - \varnothing]$ modulo the ideals $\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle$ and $\langle \sum_{F\supseteq a} z_F \mid a \in \mathfrak{A}_M\rangle$, where $\mathscr{L}_M$ denotes the lattice of flats of M and $\mathfrak{A}_M$ the set of atoms in $\mathscr{L}_M$.

In [2], Backman, Eur, and Simpson introduced a set of generators for $A^\bullet(M)$ called the *simplicial generators* (independently defined by Yuzvinsky [13] in the context of linear subspace arrangement complements). The simplicial generators are defined, for each nonempty flat F of M, by $h_F(M) = -\sum_{G\supseteq F} z_G(M) \in A^1(M)$. We write h_F for $h_F(M)$ when M is clear from context. We denote by $A^\bullet_\nabla(M)$ the presentation of the Chow ring of M by the simplicial generators:

$$A^\bullet_\nabla(M) = \mathbb{R}[h_F \mid F \in \mathscr{L}_M - \varnothing]/(I + J),$$

where $I = \langle h_a \mid a \in \mathfrak{A}_M\rangle$ and $J = \langle (h_F - h_{F\vee F'})(h_{F'} - h_{F\vee F'}) \mid F, F' \in \mathscr{L}_M - \varnothing\rangle$. We note that this presentation of J, appearing in [9], differs from that of [2].

In the *free matroid* on E, all subsets $A \subseteq E$ are flats. The Chow ring of the free matroid surjects onto $A^\bullet_\nabla(M)$, for any loopless matroid M on E, by $h_A \mapsto h_F$, where F is the closure in M of a given nonempty subset A of E. In what follows, we simply write $h_A(M)$, or h_A, for $h_{\mathrm{cl}_M(A)}(M)$. The simplicial presentation lends itself to the following combinatorial interpretation of the Chow ring of M. Let A be a nonempty subset of E with $\mathrm{rk}_M(A) > 1$ (if $\mathrm{rk}_M(A) = 1$, then $h_A = 0$ by definition). The simplicial generator h_A corresponds (via a combinatorial analogue of the cap product) to the *principal truncation* of M at the flat $F = \mathrm{cl}_M(A)$, denoted $T_F(M)$; this is the matroid with bases $B - f$ over all bases B of M which intersect F nontrivially, and over all $f \in B \cap F$ [2, Theorem 3.2.3]. Furthermore, the cap product allows for a bijection between the monomial basis for $A^c_\nabla(M)$, $\left\{h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \mid \sum a_i = c,\ \varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k,\ 1 \le a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1})\right\}$, and a class of matroids called *loopless relative nested quotients of* M. We will forego the definition of these matroids until after defining their generalizations which appear in Theorem 2.1.

Let us recall some basic definitions and notations. For a matroid M on ground set E, we let $\mathcal{I}(M)$, $\mathcal{B}(M)$, and $\mathcal{C}(M)$ denote the collections of independent sets, bases, and circuits of M, respectively. The rank in M of a subset S of E is denoted by $\mathrm{rk}_M(S)$. The *uniform matroid* $U_{k,E}$ is the matroid whose bases are all of the k-subsets of E. The *dual* of M, denoted M^*, is the matroid on E whose bases are the complements of the bases of M. The *Bergman fan* Σ_M of M is the polyhedral fan in $\mathbb{R}^E/\langle e_E\rangle$ consisting of the cones $\mathrm{cone}\{e_F \mid F \in \mathcal{F}\}$ for each flag $\mathcal{F} = \{\varnothing \neq F_1 \subsetneq \cdots \subsetneq F_t \neq E\}$ of flats of M, where e_F denotes the indicator vector for F. For a matroid M of rank $d+1$, up to scaling, there is a unique d-dimensional Minkowski weight on Σ_M [1], which is known as the *Bergman class* Δ_M *of* M.

*Corresponding author (calum.buchanan@uvm.edu).

For a nonempty subset S of E, let H_S denote the corank-1 matroid on E with collection of bases $\{E - s \mid s \in S\}$. It is well-known that the Bergman fans of corank-1 matroids are precisely the tropical hyperplanes centered at the origin. In [12], Speyer defined a notion of stable intersection for tropical linear spaces. This relates closely to matroid intersection, defined as follows. The *matroid intersection* of matroids M and N on a common ground set E, denoted $M \wedge N$, is the matroid whose spanning sets are the intersections of the spanning sets of M and N. It is noted in [7] that, as a special case of Theorem 4.11 in [12], the stable intersection of Bergman fans Σ_M and Σ_N is precisely $\Sigma_{M \wedge N}$ whenever $M \wedge N$ is loopless. A related notion is that of *matroid union*, defined in [10]: $M \vee N$ is the matroid whose independent sets are of the form $I \cup J$ for $I \in \mathcal{I}(M)$ and $J \in \mathcal{I}(N)$. We will make use of the fact that $M \wedge N$ can equivalently be defined as $(M^* \vee N^*)^*$.

We characterize the products of simplicial generators in $A^\bullet_\nabla(M)$ using the duals of certain Rado matroids. In order to define a Rado matroid, we first recall that a *transversal* of a collection $\{X_1, \ldots, X_m\}$ of (not necessarily distinct) nonempty subsets of a finite set Y is a set $\{y_1, \ldots, y_m\}$ of distinct elements in Y such that $y_i \in X_i$ for all i.

Theorem 1.1 (Rado's theorem [11]). *Let M be a matroid on Y, and let $\mathcal{X}$ be a collection of subsets $X_1, \ldots, X_m$ of Y. There exists a transversal of $\mathcal{X}$ which is independent in M if and only if $\mathrm{rk}_M(\cup_{j \in J} X_j) \geq |J|$ for all $J \subseteq \{1, \ldots, m\}$.*

Using Rado's theorem, it is not hard to show that the subsets of $\mathcal{X}$ which have independent transversals in M form the independent sets of a matroid on $\mathcal{X}$.

Definition 1.1. *For a matroid M on Y and a collection $\mathcal{X}$ of subsets $X_1, \ldots, X_m$ of Y, the* Rado matroid induced by $\mathcal{X}$ and M *is a matroid with ground set $\mathcal{X}$. Its independent sets are given by the subcollections of $\mathcal{X}$ with a transversal in $\mathcal{I}(M)$.*

Abusing terminology slightly, given a bipartite graph H with ordered partition $(\mathcal{X}, Y)$, the subsets of $\mathcal{X}$ which can be matched to a set in $\mathcal{I}(M)$ form the independent sets of the *Rado matroid induced by H and M*, denoted $R_{H,M}$. The ground set of $R_{H,M}$ will always be the first part of the partition $(\mathcal{X}, Y)$ of H.

When M is the free matroid on Y, Rado's theorem specializes to Hall's theorem on transversals, and we obtain transversal matroids from Rado matroids.

2. Main result

We first introduce a class of graphs to be used in our characterization of products of simplicial generators in $A^\bullet_\nabla(M)$.

Definition 2.1. *Let $\mathcal{A}$ be a collection of nonempty subsets $A_1, \ldots, A_m$ of a finite set E, and let $\hat{E}$ be a copy of E, $\hat{E} = \{\hat{e} \mid e \in E\}$. We define $G(\mathcal{A})$ to be the graph with bipartition $(E, \hat{E} \cup \mathcal{A})$ and edge set*

$$\{e\hat{e} \mid e \in E\} \cup \{eA \mid A \in \mathcal{A} \textit{ and } e \in A\}.$$

Figure 2.1 depicts an example of such a graph. We remind the reader that $G(\mathcal{A})$ is *not* the graph typically used to represent the set system $\mathcal{A}$, which was denoted by H in Definition 1.1.

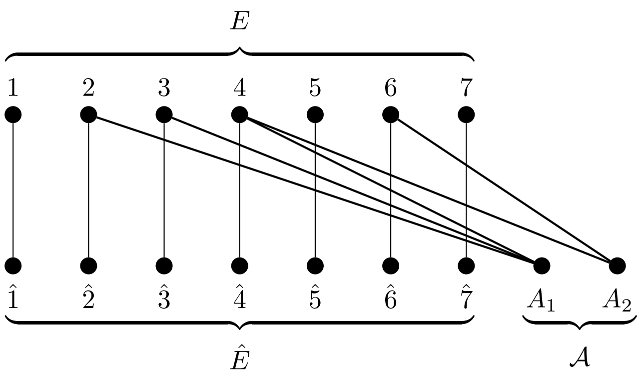

Figure 2.1: The graph $G(\mathcal{A})$, where $E = \{1, \ldots, 7\}$, $\mathcal{A} = \{A_1, A_2\}$, $A_1 = \{2, 3, 4\}$, and $A_2 = \{4, 6\}$.

Theorem 2.1. *Let $\mathcal{A}$ be a collection of (not necessarily distinct) nonempty subsets $A_1, \ldots, A_m$ of a finite set E, let M be a matroid on E, and let $G = G(\mathcal{A})$. We have*

$$M \wedge H_{A_1} \wedge \cdots \wedge H_{A_m} = (R_{G,N})^*,$$

where N is the matroid $\hat{M}^* \oplus U_{m,\mathcal{A}}$ on $\hat{E} \cup \mathcal{A}$, and $\hat{M}^*$ is a copy of M^* on $\hat{E}$.

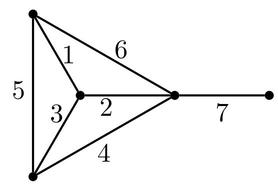

Figure 2.2: A graph whose edge set is the set E in Figure 2.1 and whose graphic matroid is not a strict gammoid.

Before proving Theorem 2.1, we note an important corollary. Namely, that the matroids $(R_{G,N})^*$ in Theorem 2.1, when they are loopless, are in natural bijection with the nonzero products of simplicial generators in $A^\bullet_\nabla(M)$. In fact, we can determine precisely which of these matroids are loopless using the *Dragon Hall-Rado condition of* M: for a matroid M on E, a collection $\mathcal{A}$ of subsets $A_1, \ldots, A_m$ of E is said to satisfy $\mathrm{DHR}(M)$ if $\mathrm{rk}_M(\cup_{j\in J} A_j) > |J|$ for any nonempty subset J of $\{1, \ldots, m\}$. Equivalently, $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$ if and only if, for every $e \in E$, there is a transversal $I \subseteq E - e$ of $\mathcal{A}$ which is independent in M [2, Proposition 5.2.3]. That $R_{G(\mathcal{A}),N}$ has a coloop whenever $\mathcal{A}$ does not satisfy $\mathrm{DHR}(M)$ is clear from this equivalent definition, for if there is an element $e \in E$ which is in every independent transversal of $\mathcal{A}$, then it is in every basis of $R_{G(\mathcal{A}),N}$.

Corollary 2.1. *Let $\mathcal{A}$ be a collection of nonempty subsets $A_1, \ldots, A_m \subseteq E$, and let M be a loopless matroid on E. The product of simplicial generators $h_{A_1} \cdots h_{A_m}$ in $A^\bullet_\nabla(M)$ is the Bergman class of the matroid $(R_{G(\mathcal{A}),N})^*$ from Theorem 2.1 whenever $\mathcal{A}$ satisfies* $\mathrm{DHR}(M)$.

Proof of Corollary 2.1 assuming Theorem 2.1. We have noted the equivalence of a stable intersection of Bergman fans and the Bergman fan of a matroid intersection whenever the matroid intersection is loopless. We now show that, whenever $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$, the matroid $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_m}$ is loopless of rank $\mathrm{rk}_M(E) - m$. The result then follows form a direct application of Theorem 2.1.

We proceed by induction on $|\mathcal{A}|$. When $\mathcal{A} = \{A\}$, this is given by [2, Theorem 3.2.3]. Now, assuming $|\mathcal{A}| \geq 2$, let $A \in \mathcal{A}$, and let $e \in E$. Further, let $I \subseteq E - e$ be a transversal of $\mathcal{A}$ which is independent in M. Note that $E - I$ is a spanning set for M^*. If a denotes the representative for A in I, then $(E - I) \cup a$ spans $M^* \vee H^*_A$. Thus, $I - a$ is an independent set in $M \wedge H_A$. Since $I - a$ is a transversal of $\mathcal{A} - A$ which avoids the arbitrarily chosen element $e \in E$, we have that $\mathcal{A} - A$ satisfies $\mathrm{DHR}(M \wedge H_A)$. An application of the inductive hypothesis to the loopless matroid $M \wedge H_A$ of rank $\mathrm{rk}_M(E) - 1$ completes the proof. □

Proof of Theorem 2.1. We work by induction on m. Since $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_m} = (M^* \vee H^*_{A_1} \vee \cdots \vee H^*_{A_m})^*$, it suffices to show that $M^* \vee H^*_{A_1} \vee \cdots \vee H^*_{A_m} = R_{G,N}$. The base case, $m = 0$, is trivial. Let $m \geq 1$, and assume that the result holds for the collection $\mathcal{A} - A_m$; that is,

$$R' := M^* \vee H^*_{A_1} \vee \cdots \vee H^*_{A_{m-1}}$$

is the Rado matroid on E induced by $G(\mathcal{A} - A_m)$ and $\hat{M}^* \oplus U_{m-1,\mathcal{A}-A_m}$. It suffices to show that the independent sets of $R' \vee H^*_{A_m}$ are precisely the independent sets of $R_{G,N}$.

First, suppose that a subset I of E is independent in $R' \vee H^*_{A_m}$. We will show that I is matched in G to an independent set in the matroid $\hat{M}^* \oplus U_{m,\mathcal{A}}$ on N. By definition, $I = J \cup K$, where $J \in \mathcal{I}(R')$ and $K \in \mathcal{I}(H^*_{A_m}) = \mathcal{I}(U_{1,A_m})$. If $K \subseteq J$, then I is independent in R', and thus I is independent in $R_{G,N}$. Otherwise, we have $K = \{a\}$ for some $a \notin J$. Take a matching from J to an independent set of R' and add the edge aA_m, which is clearly disjoint from the others, to obtain a matching from I to an independent set in $R_{G,N}$.

Second, suppose that $I \subseteq E$ is matched in G to a subset L of $\hat{E} \cup \mathcal{A}$ which is independent in $\hat{M}^* \oplus U_{m,\mathcal{A}}$. If $A_m \notin L$, then I is matched in G to an independent set in $\hat{M}^* \oplus U_{m-1,\mathcal{A}-A_m}$ on $\hat{E} \cup \{A_1, \ldots, A_{m-1}\}$; that is, $I \in \mathcal{I}(R')$, and thus $I \in \mathcal{I}(R' \vee H^*_{A_m})$. Otherwise, if $A_m \in L$, then there exists some a in $A_m \cap I$ such that $I - a$ is matched to an independent set in $\hat{M}^* \oplus U_{m-1,\mathcal{A}-A_m}$ on $\hat{E} \cup \{A_1, \ldots, A_{m-1}\}$. Thus, $I - a \in \mathcal{I}(R')$, which implies that $I \in \mathcal{I}(R' \vee H^*_{A_m})$. We have shown that $\mathcal{I}(R' \vee H^*_{A_m}) = \mathcal{I}(R_{G,N})$, which completes the proof. □

Example 2.1. *Let us again consider the graph $G = G(\mathcal{A})$ in Figure 2.1. Let M be the graphic matroid for the graph with edge set E depicted in Figure 2.2. The matroid M has rank 4. Letting $N = \hat{M}^* \oplus U_{2,\mathcal{A}}$ and $R = R_{G,N}$, we have $M \wedge H_{A_1} \wedge H_{A_2} = R^*$ by Theorem 2.1. The matroid R^* is a rank-2 matroid with set of bases $\{17, 27, 37, 47, 57, 67\}$. For instance, $\{1, 2, 3, 5, 6\} \subset E$ is matched in G to $\{\hat{1}, \hat{3}, \hat{5}, A_1, A_2\} \in \mathcal{B}(M^* \oplus U_{2,\mathcal{A}})$, and so $\{4, 7\} \in \mathcal{B}(R^*)$.*

The duals of transversal matroids are known as *strict gammoids*. We refer to the dual of a Rado matroid as a *coRado matroid*. As we noted earlier, the Bergman fans of corank-1 matroids are precisely the tropical hyperplanes centered at the origin. Thus, Theorem 2.1 implies that the stable intersection of a Bergman fan Σ_M with a collection of tropical hyperplanes centered at the origin is the Bergman fan of a coRado matroid $(R_{G,N})^*$. Letting M be the free matroid, we recover a special case of a theorem of Fink and Olarte.

Corollary 2.2 (see [6]). *A matroid is a strict gammoid if and only if its Bergman fan is a stable intersection of tropical hyperplanes centered at the origin.*

Theorem 7.5 of [6] states, more generally, that a valuated matroid is a valuated strict gammoid if and only if its associated tropical linear space is a stable intersection of tropical hyperplanes; Corollary 2.2 is the special case in which the valuations are trivial. The graphic matroid for the graph in Figure 2.2, however, is not a strict gammoid, and thus the Bergman fan of the coRado matroid R^* in Example 2.1 is not a stable intersection of tropical hyperplanes.

We now return to the loopless relative nested quotients of a matroid M, which are shown in [2] to be in correspondence with the monomial bases for the graded pieces of $A^\bullet_\nabla(M)$. First, we note that any principal truncation $T_F(M)$ is given by the dual of the Rado matroid induced by the graph $G(\{F\})$ and the matroid $\hat{M}^* \oplus U_{1,\{F\}}$ on $\hat{E} \sqcup \{F\}$. Now, we recall that the monomial basis for the graded piece of degree c, $A^c_\nabla(M)$, is the set of products $h_{F_1}^{a_1} \cdots h_{F_m}^{a_m}$ of simplicial generators corresponding to nested nonempty flats $F_1, \ldots, F_m \in \mathscr{L}_M$, with each $1 \le a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1})$ and $\sum a_i = c$. The coRado matroids in Theorem 2.1 provide a new definition for the *relative nested quotients of* M: they are matroids of the form $(R_{G(\mathcal{A}),N})^*$, where $\mathcal{A}$ is a multiset of nested flats as described above, with a_i copies of F_i for each i.

The graphs $G(\mathcal{A})$ in Definition 2.1 and the Rado matroids associated to them in Theorem 2.1 can also be used to provide an alternate proof of the Dragon Hall-Rado theorem of [2]. Coincidentally, Larson also obtained an alternate proof in [8], posted the day before the first preprint of this paper was made available.

Corollary 2.3 (Dragon Hall-Rado theorem [2]). *Let $A_1, \ldots, A_d$ be nonempty subsets of a finite set E, and let M be a loopless matroid on E of rank $d+1$. We have $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_d} = U_{1,E}$ if and only if $\{A_1, \ldots, A_d\}$ satisfies* $\mathrm{DHR}(M)$.

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_d\}$, let $G = G(\mathcal{A})$, and let $R = R_{G,N}$, where N is the matroid $\hat{M}^* \oplus U_{d,\mathcal{A}}$ on $\hat{E} \cup \mathcal{A}$ (as in Theorem 2.1). We recall that $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$ if and only if, for any $e \in E$, there is a transversal I of $\mathcal{A}$ such that $e \notin I$ and $I \in \mathcal{I}(M)$. Thus, it suffices to check that $R = U_{|E|-1,E}$ if and only if, for any $e \in E$, $\mathcal{A}$ is matched in G to an independent set in M which does not contain e.

To prove sufficiency, we recall from the proof of Corollary 2.1 that, when $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$, the coRado matroid R^* is loopless of rank $d+1-d$. To prove necessity, suppose that $R = U_{|E|-1,E}$, and let $e \in E$ be arbitrary. Since $E - e$ is a basis of R, it is matched in G to a basis $\hat{B}^*$ of $\hat{M}^*$ (of cardinality $|E| - d - 1$) and to each of the vertices $A_1, \ldots, A_d \in V(G)$. Letting I be the set of vertices matched to $A_1, \ldots, A_d$, we see that I is independent in M and does not contain e. This completes the proof. □

Eur and Larson [4] generalized the simplicial presentation for the Chow ring of a matroid to augmented Chow rings of polymatroids. We expect that Theorem 2.1 generalizes to the case of polymatroids as well. We welcome interested researchers to explore this direction.

Acknowledgments

The authors would like to thank Spencer Backman for valuable advice and conversations during the project. Spencer Backman would in turn like to thank Chris Eur, Alex Fink, Jorge Olarte, Benjamin Schröter, and Connor Simpson for inspiring discussions. The authors received helpful suggestions from Chris Eur and Connor Simpson as well. Finally, the authors thank the anonymous referees for their significant suggestions, which added clarity to this article.

References

[1] K. Adiprasito, J. Huh, E. Katz. Hodge theory for combinatorial geometries, *Ann. Math.* **188(2)** (2018) 381–452.
[2] S. Backman, C. Eur, C. Simpson, Simplicial generation of Chow rings of matroids, *J. Eur. Math. Soc.*, In press, DOI: 10.4171/JEMS/1350.
[3] C. De Concini, C. Procesi, Wonderful models of subspace arrangements, *Selecta Math.* **1** (1995) 459–494.
[4] C. Eur, M. Larson, Intersection theory of polymatroids, *Int. Math. Res. Not. IMRN* **2024(5)** (2024) 4207–4241.
[5] E. M. Feichtner, S. Yuzvinsky, Chow rings of toric varieties defined by atomic lattices, *Invent. Math.* **155(3)** (2004) 515–536.
[6] A. Fink, J. A. Olarte, Presentations of transversal valuated matroids, *J. Lond. Math. Soc.* **105(1)** (2022) 24–62.
[7] S. Hampe, The intersection ring of matroids, *J. Combin. Theory Ser. B* **122** (2017) 578–614.
[8] M. Larson, Straightening laws for Chow rings of matroids, *arXiv*:2402.03444 [math.CO], (2024).
[9] M. Larson, S. Li, S. Payne, N. Proudfoot, K-rings of wonderful varieties and matroids, *Adv. Math.* **441** (2024) #109554.
[10] C. S. J. A. Nash-Williams, An application of matroids to graph theory, In: *Theory of Graphs, International Symposium*, Rome, 1966, 263–265.
[11] R. Rado, A theorem on independence relations, *Q. J. Math.* **13** (1942) 83–89.
[12] D. E. Speyer, Tropical linear spaces, *SIAM J. Discrete Math.* **22(4)** (2008) 1527–1558.
[13] S. Yuzvinsky, Small rational model of subspace complement, *Trans. Amer. Math. Soc.* **354(5)** (2002) 1921–1945.